ON A LOCAL CHARACTERIZATION OF PSEUDOCONVEX DOMAINS

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ABSTRACT. Pseudoconvexity of a domain in $\mathbb{C}^n$ is described in terms of the existence of a locally defined plurisubharmonic/holomorphic function near any boundary point that is unbounded at the point.

1. Introduction and results

It is well-known that a domain $D \subset \mathbb{C}^n$ is pseudoconvex if and only if any of the following conditions holds:

(i) there is a smooth strictly plurisubharmonic function u on D with $\lim_{z\to\partial D} u(z) = \infty$;

(ii) for any $a \in \partial D$ there is a $u_a \in \mathcal{PSH}(D)$ with $\lim_{z\to a} u_a(z) = \infty$;

(iii) there is an $f \in \mathcal{O}(D)$ such that for any $a \in \partial D$ and any neighborhood U_a of a one has that $\limsup_{G\ni z\to a} |f(z)| = \infty$ for any connected component G of $D \cap U_a$ with $a \in \partial G$;

(iv) for any $a \in \partial D$ there is a neighborhood U_a of a and an $f_a \in \mathcal{O}(D \cap U_a)$ such that for any neighborhood $V_a \subset U_a$ of a and any connected component G of $D\cap V_a$ with $a \in \partial G$ one has $\limsup_{G\ni z\to a} |f_a(z)| = \infty$ (see Corollary 4.1.26 in [2]).

If D is C^1-smooth, we may assume that $D \cap U_a$ is connected in (iii) and (iv).

Our first aim is to see that in (i) in general 'lim' cannot be weakened by 'limsup' even if D is C^1-smooth.

Theorem 1. *For any $\varepsilon \in (0, 1)$ there is a non-pseudoconvex bounded domain $D \subset \mathbb{C}^2$ with $C^{1,1-\varepsilon}$-smooth boundary and a negative function $u \in \mathcal{PSH}(D)$ with $\limsup_{z\to a} u(z) = 0$ for any $a \in \partial D$.*

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In particular, $v := -\log(-u) \in \mathcal{PSH}(D)$ *with* $\limsup_{z\to a} v(z) = \infty$ *for any* $a \in \partial D$.

If we do not require smoothness of D, following the idea presented in the proof, we may just take $D = \{z \in \mathbb{C}^n : \min\{||z||, ||z-a||\} < 1\}$, $0 < ||a|| < 2$, $n \geq 2$.

On the other, this cannot happen if D is C^2-smooth.

Proposition 2. *Let* $D \subset \mathbb{C}^n$ *be a* C^2*-smooth domain with the following property: for any boundary point* $a \in \partial D$ *there is a neighborhood* U_a *of* a *and a function* $u_a \in \mathcal{PSH}(D \cap U_a)$ *such that* $\limsup_{z\to a} u_a(z) = \infty$. *Then* D *is pseudoconvex.*

However, if we replace 'limsup' by 'lim', we may remove the hypothesis about smoothness of the boundary.

Proposition 3. *Let* $D \subset \mathbb{C}^n$ *be a domain with the following property: for any boundary point* $a \in \partial D$ *there is a neighborhood* U_a *of* a *and a function* $u_a \in \mathcal{PSH}(D \cap U_a)$ *such that* $\lim_{z\to a} u_a(z) = \infty$. *Then* D *is pseudoconvex.*

Note that the assumption in Proposition 3 is formally weaker that to assume that D is locally pseudoconvex.

Remark. The three propositions above have real analogues replacing (non)pseudoconvex domains by (non)convex domains and plurisubharmonic functions by convex functions (for the analogue of Proposition 3 use e.g. Theorem 2.1.27 in [2] which implies that if D is a nonconvex domain in $\mathbb{R}^n$, then there exists a segment $[a, b]$ such that $c = \frac{a+b}{2} \in \partial D$ but $[a, b] \setminus \{c\} \subset D$). The details are left to the reader.

Recall now that a domain $D \subset \mathbb{C}^n$ is called *locally weakly linearly convex* if for any boundary point $a \in \partial D$ there is a complex hyperplane H_a through a and a neighborhood U_a of a such that $H_a \cap D \cap U_a = \varnothing$. D. Jacquet asked whether a locally weakly linearly convex domain is already pseudoconvex (see [5], page 58). The answer to this question is affirmative by Proposition 3. The next proposition shows that such a domain has to be even taut[1] if it is bounded.

Proposition 4. *Let* $D \subset \mathbb{C}^n$ *be a bounded domain with the following property: for any boundary point* $a \in \partial D$ *there is a neighborhood* U_a *of* a *and a function* $f_a \in \mathcal{O}(D \cap U_a)$ *such that* $\lim_{z\to a} |f_a(z)| = \infty$. *Then* D *is taut.*

[1]This means that $\mathcal{O}(\mathbb{D}, D)$ is a normal family, where $\mathbb{D} \subset \mathbb{C}$ is the open unit disc. Note that any taut domain is pseudoconvex and any bounded pseudoconvex domain with C^1-smooth boundary is taut.

Let $D \subset \mathbb{C}^n$ be a domain and let $K_D(z)$ denote the Bergman kernel of the diagonal. It is well-known that $\log K_D \in \mathcal{PSH}(D)$. Recall that

(v) if D is bounded and pseudoconvex, and $\limsup_{z\to a} K_D(z) = \infty$ for any $a \in \partial D$, then D is an L^2_h-domain of holomorphy ($L^2_h(D) := L^2(D) \cap \mathcal{O}(D)$) (see [6]).

We show that the assumption of pseudoconvexity is essential.

Proposition 5. *There is a non-pseudoconvex bounded domain $D \subset \mathbb{C}^2$ such that $\limsup_{z\to a} K_D(z) = \infty$ for any $a \in \partial D$.*

Note that the domain D with $u = \log K_D$ presents a similar kind of example as that in Proposition 1 (however, the domain has weaker regularity properties).

The example given in Proposition 5 is a domain with non-schlicht envelope of holomorphy. This is not accidental as the following result shows.

Proposition 6. *Let $D \subset \mathbb{C}^n$ be a domain such that $\limsup_{z\to a} K_D(z) = \infty$ for any $a \in \partial D$. Assume that one of the following conditions is satisfied:*

– the envelope of holomorphy $\hat{D}$ of D is a domain in $\mathbb{C}^n$,

– for any $a \in \partial D$ and for any neighborhood U_a of a there is a neighborhood $V_a \subset U_a$ of a such that $V_a \cap D$ is connected (this is the case when e.g. D is a C^1-smooth domain).

Then D is pseudoconvex.

Remark. Note that the domain in the example is not fat. We do not know what will happen if D is assumed to be fat.

Making use of the reasoning in [3] we shall see how Proposition 5 implies that the domain from this proposition admits a function $f \in L^2_h(D)$ satisfying the property $\limsup_{z\to a} |f(z)| = \infty$ for any $a \in \partial D$.

Theorem 7. *Let D be the domain from Proposition 5. Then there is a function $f \in L^2_h(D)$ such that $\limsup_{z\to a} |f(z)| = \infty$ for any $a \in \partial D$.*

2. Proof of Proposition 1

First, we shall prove two lemmas.

Lemma 8. *For any $\varepsilon \in (0,1)$ and C_1, $C_2 > 0$, there exists an $F \in \mathcal{C}^{1,1-\varepsilon}(\mathbb{R})$ such that:*

(i) $\operatorname{supp} F \subset [-1,+1]$, $0 \le F(x) \le C_1$ *for all $x \in \mathbb{R}$;*

(ii) *there is a dense open set $\mathcal{U} \subset [-1,+1]$ such that $F''(x)$ exists and $F''(x) \le -C_2 < 0$ for all $x \in \mathcal{U}$;*

(iii) *F vanishes on a Cantor subset of $[-1,+1]$.*

Proof. An elementary construction yields an even non-negative smooth function b supported on $[-3/4, +3/4]$, decreasing on $[0, 3/4]$, such that $b(x) = 1 - 4x^2$ for $|x| \leq 1/4$, $|b'(x)| \leq C_3$, $-8 \leq b''(x) \leq C_4$ for all $x \in \mathbb{R}$, where $C_3, C_4 > 0$.

For any $a, p > 0$, we set $b_{a,p}(x) := ab(x/p)$, $x \in \mathbb{R}$.

We shall construct two decreasing sequences of positive numbers $(a_n)_{n\geq 0}$ and $(p_n)_{n\geq 0}$, and intervals $\{I_{n,i}, J_{n,i}, n \geq 0, 1 \leq i \leq 2^n\}$.

Set $I_{0,1} := (-1, +1)$ and $J_{0,1} := [-p_0/4, p_0/4]$, where $p_0 < 1$. Then $I_{1,1} := (-1, -p_0)$ and $I_{1,2} := (p_0, 1)$.

In general, if the intervals of the n-th "generation" $I_{n,i}$ are known, we require

$$p_n < \frac{|I_{n,i}|}{2}, \tag{1}$$

where $|J|$ denotes the length of an interval J. Denote by $c_{n,i}$ the center of $I_{n,i}$ and put $J_{n,i} := [c_{n,i} - p_n/4, c_{n,i} + p_n/4]$. Denote respectively by $I_{n+1,2i-1}$ and $I_{n+1,2i}$ the first and second component of $I_{n,i} \setminus J_{n,i}$.

Now we write

$$f_n(x) := \sum_{i=1}^{2^n} b_{a_n,p_n}(x - c_{n,i}),\ x \in \mathbb{R}, \quad F_n := \sum_{m=0}^{n} f_m.$$

Note that the terms in the sum defining f_n have disjoint supports contained in $[c_{n,i} - 3p_n/4, c_{n,i} + 3p_n/4] \subset I_{n,i}$, ($J_{n,i}$ does not contain the support of the corresponding term in f_n; it is only a place, where that term coincides with a quadratical polynomial) so that $|f_n'(x)| \leq C_3 a_n/p_n$. The function $F = \lim_{n\to\infty} F_n$ will be of class $\mathcal{C}^1$ if

$$\sum_{n=0}^{\infty} \frac{a_n}{p_n} < \infty. \tag{2}$$

Also, note that

$$|F_n''(x)| \leq |F_{n-1}''(x)| + C_4 \frac{a_n}{p_n^2}, \text{ so } \sup |F_n''| \leq C_4 \sum_{m=1}^{n} \frac{a_m}{p_m^2}.$$

From now on we choose

$$\frac{a_n}{p_n^2} = BA^n, \text{ for some } A > 1, B > 0 \text{ to be determined.} \tag{3}$$

We then have $\sup |F_n''| \leq C_4 BA^{n+1}/(A-1)$.

All the successive terms $f_m, m > n$, are supported on intervals of the form $I_{m,j}$, thus vanish on the interval $J_{n,i}$, so on those intervals F is a

smooth function and

$$F'' = F''_n = F''_{n-1} - 8\frac{a_n}{p_n^2} \leq C_4 \frac{BA^n}{A-1} - 8BA^n;$$

therefore, if we choose

$$A > 1 + \frac{C_4}{4}, \tag{4}$$

we have $F''(x) \leq -4BA^n$ for all $x \in J_{n,i}$, and $1 \leq i \leq 2^n$.

Set $\mathcal{U} := \bigcup_{n,i} J^\circ_{n,i}$. We have seen that $|I_{n+1,i}| < |I_{n,j}|/2$ (and those quantities do not depend on i or j), so that the complement of $\mathcal{U}$ has empty interior. This proves claim (ii), by choosing $B = C_2/4$. The other claims are clear from the form of the function F, once we provide the sequences (a_n) and (p_n) satisfying (3), (4), (2), and (1).

Let $a_n := a_0\gamma^n$, $p_n = p_0\delta^n$. Then (3) is satisfied by construction and $a_0 = Bp_0^2$. Fix $\delta, p_0 \in (0, 1/2)$. It follows that $p_n < |I_{n,i}|/4$ for all n (by an easy induction). Hence, (1) holds.

By our explicit form, (4) means that $\gamma\delta^{-2} > 1 + \frac{C_4}{4}$, while (2) means $\gamma\delta^{-1} < 1$, so with $\delta^{-1} > 1 + \frac{C_4}{4}$, it is easy to choose γ. Finally $\|F\|_\infty \leq a_0(1-\gamma)^{-1} < C_1$ for a_0 small enough, which can be achieved by decreasing p_0 further.

Given any $\varepsilon > 0$, we can modify the choices of δ and γ to obtain that $F' \in \Lambda_{1-\varepsilon}$ (the Hölder class of order $1-\varepsilon$). Given any two points $x, y \in [-1, +1]$ and any integer $n \geq 1$,

$$|F'(x) - F'(y)| \leq |x-y| \|F''_n\|_\infty + 2\sum_{m \geq n} \|f'_m\|_\infty$$

$$\leq C\left((\gamma\delta^{-2})^n |x-y| + (\gamma\delta^{-1})^n\right),$$

where $C > 0$ is a positive constant depending on the parameters we have chosen. Take n such that $\delta|x-y| \leq \delta^n \leq |x-y|$. Then

$$\frac{|F'(x) - F'(y)|}{|x-y|^{1-\varepsilon}} \leq C'(\gamma\delta^{-2+\varepsilon})^n,$$

and it will be enough to choose δ and γ so that $\gamma\delta^{-2+\varepsilon} \leq 1$ and $\gamma\delta^{-2} > 1 + \frac{C_4}{4}$, which can be achieved once we pick δ small enough. The rest of the parameters are then chosen as above. □

Remark. It is clear that F cannot be of class $\mathcal{C}^2(\mathbb{R})$. We do not know if our argument can be pushed to get $F \in \mathcal{C}^{1,1}(\mathbb{R})$.

Lemma 9. *For any $\varepsilon \in (0,1)$ there exists a non-pseudoconvex bounded $\mathcal{C}^{1,1-\varepsilon}$-smooth domain $D \subset \mathbb{C}^2$ boundary such that ∂D contains a dense subset of points of strict pseudoconvexity.*

Proof. We start with the unit ball and cave it in somewhat at the North Pole to get an open set of points of strict pseudoconcavity on the boundary. Let $r_0 < 1/3$ and for $x \in [0,1)$,

$$\psi_0(x) = \min\{\log(1-x^2), x^2 - r_0^2\}.^2$$

We take ψ a $\mathcal{C}^\infty$ regularization of ψ_0 such that $\psi = \psi_0$ outside of $(r_0/2, r_0)$. Consider the Hartogs domain

$$D_0 := \left\{(z,w) \in \mathbb{C}^2 : |z| < 1, \log|w| < \frac{1}{2}\psi(|z|)\right\}.$$

Notice that $D_0 \setminus \{|z| \le r_0\} = \mathbb{B}_2 \setminus \{|z| \le r_0\}$, so that ∂D is smooth near $|z| = 1$.

Now define $\Phi(z) = \Phi(x+iy) = F(x/r_0)\chi(y/r_0)$, where F is the function obtained in Lemma 8, and χ is a smooth, even cut-off function on $\mathbb{R}$ such that $0 \le \chi \le 1$, $\operatorname{supp}\chi \subset (-2,2)$, and $\chi \equiv 1$ on $[-1,1]$. We define

$$D := \left\{(z,w) \in \mathbb{C}^2 : |z| < 1, \log|w| < \frac{1}{2}\psi(|z|) + \Phi(z)\right\}.$$

Recall that for a Hartogs domain $\{\log|w| < \varphi(z), |z| < 1\}$, if φ is of class $\mathcal{C}^2$ at z_0, a boundary point (z_0, w_0) with $|z_0| < 1$ is strictly pseudoconvex (respectively, strictly pseudoconcave) if and only if $\Delta\varphi(z_0) < 0$ (respectively, $\Delta\varphi(z_0) > 0$). Choosing an appropriate regularization (convolution by a smooth positive kernel of small enough support), we may get that:

- $\Delta\psi(|z|) \le -4$ for $|z| \ge r_0$,
- $\Delta\psi(|z|) = 4$ for $|z| \le r_0/2$, and is always ≤ 4.

We consider points $z_0 = x + iy$. If $|x| > r_0$, $\Phi(z_0) = 0$ and we have pseudoconvex points (the boundary is a portion of the boundary of the ball).

On the other hand, when $x \in r_0\mathcal{U}$ (where $\mathcal{U}$ is the dense open set defined in Lemma 8),

$$\Delta\Phi(z_0) = \frac{1}{r_0^2}\Big(F''(x/r_0)\chi(y/r_0) + F(x/r_0)\chi''(y/r_0)\Big).$$

The only values of z_0 for which $F(x/r_0)\chi''(y/r_0) \ne 0$ or $\chi(y/r_0) < 1$ verify $|z_0| > r_0$, and at those points we have, using the fact that $F''(x/r_0) < 0$,

$$\frac{1}{2}\Delta\psi(|z_0|) + \Delta\Phi(z_0) \le -4 + \frac{1}{r_0^2}C_1\|\chi''\|_\infty \le -1$$

[2] Note that the graphs of both functions cut inside the interval $(r_0/2, r_0)$. Indeed, $x^2 - r_0^2 > \log(1-x^2)$ for $x \ge r_0^2$ and $x^2 - r_0^2 < \log(1-x^2)$ for $x \le r_0^2/2$.

if we choose C_1 small enough. Hence we have strict pseudoconvexity again.

So we may restrict attention to $|y| \leq r_0$ and $\Delta\Phi(z_0) = F''(x/r_0)/r_0^2$. Therefore
$$\frac{1}{2}\Delta\psi(|z_0|) + \Delta\Phi(z_0) \leq 2 - C_2/r_0^2 < -2$$
for a C_2 chosen large enough.

Finally, notice that points (z_0, w_0) with $|z_0| < r_0/2$ and $F(x) = 0$ verify $(z_0, w_0) \in \partial D_0 \cap \partial D$, $D_0 \subset D$, and D_0 is strictly pseudoconcave at (z_0, w_0), so D is as well. $\square$

Proof of Proposition 1. Let D be the domain from Lemma 9. We may choose a dense countable subset $(a_j) \subset \partial D$ of points of strict pseudoconvexity. For any j, there is a negative function $u_j \in \mathcal{PSH}(D)$ with $\lim_{z\to a_j} u_j(z) = 0$. If (D_j) is an exhaustion of D such that $D_j \Subset D_{j+1}$ and $m_j = -\sup_{D_j} u_j$, then it is enough to take u to be the upper semicontinuous regularization of $\sup_j u_j/m_j$. $\square$

3. Proofs of Propositions 2, 3 and 4

Proof of Proposition 2. We may assume that D has a global defining function $r : U \to \mathbb{R}$ with $U = U(\partial D)$, $r \in \mathcal{C}^2(U)$, and $\operatorname{grad} r \neq 0$ on U, such that $D \cap U = \{z \in U : r(z) < 0\}$.

Now assume the contrary. Then we may find a point $z^0 \in \partial D$ such that the Levi form of r at z^0 is not positive semidefinite on the complex tangent hyperplane to ∂D at z_0. Therefore, there is a complex tangent vector a with $\mathcal{L}r(z_0, a) \leq -2c < 0$, where $\mathcal{L}r(z_0, a)$ denotes its Levi form at z^0 in direction of a. Moreover, we may assume that $|\frac{\partial r}{\partial z_1}(z_0)| \geq 2c$.

Now choose $V = V(z^0) \subset U$ and $u \in \mathcal{PSH}(D \cap V)$ with
$$\limsup_{D\cap V \ni z \to z_0} u(z) = \infty;$$
in particular, there is a sequence of points $D \cap V \ni b^j \to z_0$ such that $u(b^j) \to \infty$.

By the $\mathcal{C}^2$-smooth assumption, there is an $\varepsilon_0 > 0$ such that for all $z \in \mathbb{B}(z_0, \varepsilon_0) \subset V$ and all $\tilde{a} \in \mathbb{B}(a, \varepsilon_0)$ we have
$$\mathcal{L}r(z, \tilde{a}) \leq -c, \quad |\frac{\partial r}{\partial z_1}(z)| \geq c.$$

Now fix an arbitrary boundary point $z \in \partial D \cap \mathbb{B}(z_0, \varepsilon_0)$. Define
$$a(z) := a + (-\frac{\sum_{j=1}^n a_j \frac{\partial r}{\partial z_j}(z)}{\frac{\partial r}{\partial z_1}(z)}, 0, \ldots, 0).$$

Observe that this vector is a complex tangent vector at z and $a(z) \in \mathbb{B}(a, \varepsilon_0)$ if $z \in \mathbb{B}(z_0, \varepsilon_1)$ for a sufficiently small $\varepsilon_1 < \varepsilon_0$.

Now, let $z \in \partial D \cap \mathbb{B}(z_0, \varepsilon_1)$. Put

$$b_1(z) := \frac{\mathcal{L}r(z, a(z))}{2\frac{\partial r}{\partial z_1}(z)}$$

and

$$\varphi_z(\lambda) = z + \lambda a + (\lambda a_1(z) + \lambda^2 b_1(z), 0, \dots, 0), \quad \lambda \in \mathbb{C}.$$

Moreover, if ε_1 is sufficiently small, we may find $\delta, t_0 > 0$ such that for all $z \in \partial D \cap \mathbb{B}(z_0, \varepsilon_1)$ we have

$$\overline{D} \cap \mathbb{B}(z, \delta) - t\nu(z) \subset D, \quad 0 < t \le t_0,$$

where $\nu(z)$ denotes the outer unit normal vector of D at z.

Next using the Taylor expansion of φ_z, $z \in \partial D \cap \mathbb{B}(z_0, \varepsilon_1)$, ε_1 sufficiently small, we get

$$r \circ \varphi_z(\lambda) = |\lambda|^2 \Big(\mathcal{L}r(z, a(z)) + \varepsilon(z, \lambda) \Big),$$

where $|\varepsilon(z, \lambda)| \le \varepsilon(\lambda) \to 0$ if $\lambda \to 0$.

In particular, $\varphi_z(\lambda) \in \mathbb{B}(z, \delta) \cap D \subset V \cap D$ when $0 < |\lambda| \le \delta_0$ for a certain positive δ_0 and $r \circ \varphi_z(\lambda) \le -\delta_0^2 c/2$ when $|\lambda| = \delta_0$.

Hence, $K := \bigcup_{z \in \partial D \cap \mathbb{B}(z_0, \varepsilon_1), |\lambda| = \delta_0} \varphi_z(\lambda) \Subset D \cup V$. Choose an open set $W = W(K) \Subset D \cap V$. Then $u \le M$ on W for a positive M.

Finally, choose a j_0 such that $b^j = z^j - t_j \nu(z^j)$, $j \ge j_0$, where $z^j \in \partial D \cap \mathbb{B}(z_0, \varepsilon_1)$, $0 < t_j \le t_0$, and $\varphi_{z^j}(\lambda) \in W$ when $|\lambda| = \delta_0$. Therefore, by construction, $u(b^j) \le M$, which contradicts the assumption. $\square$

Proof of Proposition 3. Assume that D is not pseudoconvex. Then, by Corollary 4.1.26 in [2], there is $\varphi \in \mathcal{O}(\mathbb{D}, D)$ such $\operatorname{dist}(\varphi(0), \partial D) < \operatorname{dist}(\varphi(\zeta), \partial D)$ for any $\zeta \in \mathbb{D}_*$. To get a contradiction, it remains to use similar arguments as in the previous proof and we skip the details.

Proof of Proposition 4. It is enough to show that if $\mathcal{O}(\mathbb{D}, D) \ni \psi_j \to \psi$ and $\psi(\zeta) \in \partial D$ for some $\zeta \in \mathbb{D}$, then $\psi(\mathbb{D}) \subset \partial D$. Suppose the contrary. Then it is easy to find points $\eta_k \to \eta \in \mathbb{D}$ such that $\psi(\eta_k) \in D$ but $a = \psi(\eta) \in \partial D$. We may assume that $\eta = 0$ and $g_a = \frac{1}{f_a}$ is bounded on $D \cap U_a$. Let $r \in (0, 1)$ be such that $\psi(r\mathbb{D}) \Subset U_a$. Then $\psi_j(r\mathbb{D}) \subset U_a$ for any $j \ge j_0$. Hence $|g_a \circ \psi_j| < 1$ and we may assume that $g_a \circ \psi_j \to h_a \in \mathcal{O}(r\mathbb{D}, \mathbb{C})$. Since $h_a(\eta) = 0$, it follows by the Hurwitz theorem that $h_a = 0$. This contradicts the fact that $h_a(\eta_k) = g_a \circ \psi(\eta_k) \ne 0$ for $|\eta_k| < r$. $\square$

4. Proofs of Propositions 5, 6 and 7

Proof of Proposition 5. Our aim is to construct a non-pseudoconvex bounded domain $D \subset \mathbb{C}^2$ such that $\limsup_{z \to a} K_D(z) = \infty$ for any $a \in \partial D$.

Let us start with the domain $P \times \mathbb{D}$, where $P = \{\lambda \in \mathbb{C} : \frac{1}{2} < |\lambda| < \frac{3}{2}\}$. Let

$$S := \{(z_1, z_2) = (x_1 + iy_1, z_2) \in P \times \mathbb{D} : (x_1 - 1)^2 + \frac{1 + |z_2|^2}{1 - |z_2|^2} y_1^2 = \frac{1}{4}, y_1 > 0\}.$$

Define $D := (P \times \mathbb{D}) \setminus S$. Note that D is a domain. Its envelope of holomorphy is non-schlicht and consists of the union of D and one additional 'copy' of the set

$$D_1 := \{(z_1, z_2) \in P \times \mathbb{D} : (x_1 - 1)^2 + \frac{1 + |z_2|^2}{1 - |z_2|^2} y_1^2 \leq \frac{1}{4}, y_1 > 0\}.$$

In particular, D is not pseudoconvex. Note that convexity of the the interior D^0 of D_1 implies that $\lim_{z \to \partial D_1} K_{D^0}(z) = \infty$. Therefore, it follows from the localization result for the Bergman kernel due to Diederich-Fornaess-Herbort formulated for Riemann domains in the paper [4] that for all $a \in S \subset \partial D_1$ the following property holds: $\lim_{D \cap D_1 \ni z \to a} K_D(z) = \infty$ (on the other hand while tending to the points from S from the 'other side' of the domain D the Bergman kernel is bounded from above). Obviously $P \times \mathbb{D}$ is Bergman exhaustive, so for any $a \in \partial(P \times \mathbb{D})$ the following equality holds $\lim_{z \to a} K_D(z) = \infty$. □

Proof of Proposition 6. Recall the following facts that follow from [1].

If the envelope of holomorphy $\hat{D}$ of the domain D is a domain in $\mathbb{C}^n$ (is schlicht) then the Bergman kernel K_D extends to a real analytic function $\tilde{K}_D$ defined on $\hat{D}$.

Let $\emptyset \neq P_0 \subset D$, $P_0 \subset P$, $P \setminus D \neq \emptyset$ and $\bar{P}_0 \cap (\mathbb{C}^n \setminus D) \neq \emptyset$, where P_0, P are polydiscs, and the following property is satisfied: for any $f \in \mathcal{O}(D)$ there is a function $\tilde{f} \in \mathcal{O}(P)$ such that $f = \tilde{f}$ on P_0. Then the Bergman kernel K_D extends to a real analytic function on P. More precisely, there is a real analytic function $\tilde{K}_D$ defined on P such that $\tilde{K}_D(z) = K_D(z)$, $z \in P_0$.

Both facts above complete the proof of Proposition 6. □

The proof of Proposition 7 is essentially contained in [3]. However, this PhD Thesis is not publically accessible. Therefore we repeat it here. The idea is the following: if $\limsup_{z \to a} K_D(z) = \infty$ for some $a \in \partial D$, then there is an $f \in L_h^2(D)$ such that $\limsup_{z \to a} |f(z)| = \infty$.

Proof of Proposition 7. In view of Proposition 5, $\limsup_{z \to a} K_D(z) = \infty$ for any $a \in \partial D$.

Let $a \in \partial D$. We claim that there is an $L^2_h(D)$-function h which is unbounded near a.

Assume the contrary. Hence for any $f \in L^2_h(D)$ there exists a neighborhood U_f of a and a number M_f such that $|f| \leq M_f$ on $D \cap U_f$.

Denote by L the unit ball in $L^2_h(D)$ and by $c = \pi^n$.

Let $K_1 := \{z \in D : \operatorname{dist}(z, \partial D) \geq 1\}$ (if this is empty take a smaller number than 1). By the meanvalue inequality we have for any $f \in L$ that $|f| \leq c$ on K_1. By assumption, there are $z_1 \in D$ and $f_1 \in L$ such that $|z_1 - a| < 1$ and $|f_1(z_1)| > c$.

Set $g_1 := f_1/c$. Then $g \in L$ and therefore there are a neighborhood U_1 of a and number $M_1 > 1$ such that $|g_1| \leq M_1$ on $D \cap U_1$.

Set $K_2 := \{z \in D : \operatorname{dist}(z, \partial D) \geq \operatorname{dist}(z_1, \partial D)\}$ and $d = c \operatorname{dist}(z_1, \partial D)$. Then $K_1 \subset K_2$. Choose $z_2 \in U_1 \cap D$, $z_2 \notin K_2$, $|z_2 - a| < 1/2$, and $f_2 \in L$ with $|f_2(z_2)| \geq d(1^3 + 1^2 M_1)$. Moreover, $|f_2| \leq d$ on K_2. Put $g_2 := f_2/d$. Then $g_2 \in L$. Choose now a neighborhood U_2 of a and a number M_2 such that $|g_2| \leq M_2$ on $D \cap U_2$.

Then we continue this process.

So we have points $z_k \in K_{k-1}$, $z_k \notin K_{k-1}$, $|z_k - a| < 1/k$, and functions $f_k \in L$ with

$$|f_k(z_k)| \geq c \operatorname{dist}(z_{k-1}, \partial D)^n (k^3 + k^2 \sum_{j=1}^{k-1} M_j).$$

Setting $g_k := f_k/d$ and $h := \sum_{j=1}^{\infty} g_j/j^2$, it is clear that $h \in L^2_h(D)$. Fix now $k \geq 2$. Then

$$|h(z_k)| \geq \frac{|g_k(z_k)|}{k^2} - \sum_{j=1}^{k-1} \frac{|g_j(z)|}{j^2} - \sum_{j=k+1}^{\infty} \frac{|g_j(z)|}{j^2}$$

$$\geq k + \sum_{j=1}^{k-1} M_j - \sum_{j=1}^{k-1} \frac{M_j}{j^2} - \sum_{j=k+1}^{\infty} \frac{1}{j^2} > k - \frac{1}{6}.$$

In particular, h is unbounded at a which is a contradiction.

It remains to choose a dense countable sequence $(a_j) \subset \partial D$ such that any term repeats infinitely many times and to copy the proof of the Cartan-Thullen theorem. $\square$

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